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DIFFEOMORPHISMS APPROXIMATED BY ANOSOV ON THE 2-TORUS AND THEIR SBR MEASURES

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ABSTRACT. We consider the C^2 set of C^2 diffeomorphisms of the 2-torus $\mathbb{T}^2$, provided the conditions that the tangent bundle splits into the directed sum $T\mathbb{T}^2 = E^s \oplus E^u$ of Df-invariant subbundles E^s, E^u and there is $0 < \lambda < 1$ such that $\|Df|_{E^s}\| < \lambda$ and $\|Df|_{E^u}\| \geq 1$. Then we prove that the set is the union of Anosov diffeomorphisms and diffeomorphisms approximated by Anosov, and moreover every diffeomorphism approximated by Anosov in the C^2 set has no SBR measures. This is related to a result of Hu-Young.

We know that Anosov diffeomorphisms are structurally stable (Anosov [A], Robbin [R1] and Robinson [R2]) and have an SBR measure (Sinai [S1]). Recently Hu-Young [H-Y] showed that a special diffeomorphism g of the 2-torus $\mathbb{T}^2$, provided the condition that for $x \in \mathbb{T}^2$ there are $0 < \lambda < 1$ and a continuous splitting $T_x\mathbb{T}^2 = E^u_x \oplus E^s_x$ of invariant subspaces E^u_x and E^s_x such that if x is not the origin then (i) $\|Dg|_{E^s_x}\| \leq \lambda$, (ii) $\|Dg|_{E^u_x}\| > 1$, and if p is the origin then $\|Dg|_{E^u_p}\| = 1$, has no SBR measures. Such a diffeomorphism is called *almost Anosov*.

Let $\mathrm{Diff}^2(\mathbb{T}^2)$ be a set of C^2 diffeomorphisms on the 2-torus imposed with the C^2 topology. A diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is called *Anosov* if f has a hyperbolic structure on all of $\mathbb{T}^2$ (cf. [S2]).

We denote as $A(\mathbb{T}^2)$ the open set of Anosov diffeomorphisms. Then $A(\mathbb{T}^2)$ is a proper subset of the set θ^2 of diffeomorphisms such that

(iii) the tangent bundle $T\mathbb{T}^2$ splits into the directed sum $T\mathbb{T}^2 = E^u \oplus E^s$ of invariant subbundles E^u and E^s, and
(iv) there exist a C^∞ Riemannian metric $\|\cdot\|$ and $0 < \lambda < 1$ such that
$$\|D_xf|_{E^s}\| \leq \lambda, \quad \|D_xf|_{E^u}\| \geq 1.$$

Our aim is to investigate the dynamical properties of diffeomorphisms belonging to $\theta^2 \setminus A(\mathbb{T}^2)$. More precisely we state them as follows.

Theorem A. *Each diffeomorphism belonging to $\theta^2 \setminus A(\mathbb{T}^2)$ is approximated by Anosov diffeomorphisms, and it has no SBR measures.*

The conclusions will be obtained in proving the following three propositions.

Proposition B. *Let $f \in \theta^2 \setminus A(\mathbb{T}^2)$. Then the set Λ defined by*
$$\Lambda = \left\{x \in \mathbb{T}^2 \;\middle|\; \|Df^n|_{E^u_x}\| = 1 \text{ for } n \in \mathbb{Z}\right\}$$
has the following properties:

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(a) Λ *is closed, nonempty and* f*-invariant,*
(b) Λ *is expressed as the union of finite connected sets, and each connected component of* Λ *is either a single point or a* C^2 *arc which is tangential to* E^u*,*
(c) *each element of* Λ *is a periodic point, and*
(d) *there exists a* C^1 *Riemannian metric* $|||\cdot|||$ *such that*

$$|||Df|_{E^s_x}||| \le \lambda \quad (x \in \mathbb{T}^2),$$
$$|||Df|_{E^u_x}||| > 1 \quad (x \notin \Lambda) \text{ and}$$
$$|||Df|_{E^u_x}||| = 1 \quad (x \in \Lambda).$$

By Proposition B we can easily check that the following are equivalent:
(e) Λ is a finite set,
(f) f is expansive, i.e. there is a constant $e > 0$ such that $x \neq y$ $(x, y \in \mathbb{T}^2)$ implies $d(f^n(x), f^n(y)) > e$ for some integer n.

By making use of the above metric $|||\cdot|||$, we have the following:

Proposition C. *Every* $f \in \theta^2$ *is* C^2*-approximated by Anosov.*

Proposition C tells us that every $f \in \theta^2$ is homotopic to an Anosov diffeomorphism. Thus there exists a hyperbolic toral automorphism homotopic to f, and so f is semi-conjugate to the toral automorphism (see [A-H]). Then we have the following:

Corollary. *Let* $f \in \theta^2$ *and let* Λ *be as in Proposition B. Then there exist a hyperbolic toral automorphism* $A : \mathbb{T}^2 \to \mathbb{T}^2$ *and a continuous surjective map* $h : \mathbb{T}^2 \to \mathbb{T}^2$ *such that*

$$\begin{array}{ccc} \mathbb{T}^2 & \xrightarrow{\ f\ } & \mathbb{T}^2 \\ h\big\downarrow & & \big\downarrow h \\ \mathbb{T}^2 & \xrightarrow[\ A\ ]{} & \mathbb{T}^2 \end{array}$$

commutes. If Λ *is finite, then* h *is a homeomorphism.*

The second statement of the corollary follows from the fact that every expansive homeomorphism on the 2-torus is topologically conjugate to a hyperbolic toral automorphism (cf. [H]).

Proposition D. *Every* $f \in \theta^2 \setminus A(\mathbb{T}^2)$ *has no SBR measures.*

For the proof of Proposition D we need the conclusion of Proposition B and the technique in [H-Y].

Before starting the proof of Theorem A we give the notations and the definitions that we need. Let $f \in \mathrm{Diff}^2(\mathbb{T}^2)$ and μ be an f-invariant Borel probability measure of $\mathbb{T}^2$. The measure μ is called a *Sinai-Bowen-Ruelle measure* (SBR measure for abbreviation) if for μ-almost all $x \in \mathbb{T}^2$ there exist $v \in T_x\mathbb{T}^2$ and a number $\lambda(x) > 0$ satisfying

(A) $\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda(x)$,
(B) μ has a conditional measure that is absolutely continuous (with respect to the Lebesgue measure) on unstable manifolds, which is defined as follows:

If ξ is a measurable decomposition of $\mathbb{T}^2$, then a family $\{\mu^\xi_x | x \in \mathbb{T}^2\}$ of Borel probability measures exists, and it satisfies the following conditions:

(C) for x, $y \in \mathbb{T}^2$ if $\xi(x) = \xi(y)$ then $\mu_x^\xi = \mu_y^\xi$, here $\xi(x)$ denotes a set containing x in ξ,
(D) $\mu_x^\xi(\xi(x)) = 1$ for μ-almost all $x \in \mathbb{T}^2$,
(E) for any Borel set A a function $x \mapsto \mu_x^\xi(A)$ is measurable and

$$\mu(A) = \int_{\mathbb{T}^2} \mu_x^\xi(A) d\mu(x).$$

The family $\{\mu_x^\xi | x \in \mathbb{T}^2\}$ is called a *canonical system of conditional measures* for μ and ξ (see [R3] for more details).

Whenever f has the condition (A), then a set

$$W^u(x) = \left\{ y \in \mathbb{T}^2 | \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}(x), f^{-n}(y)) < 0 \right\}$$

is a *unstable manifold* for μ-almost all x in $\mathbb{T}^2$ ([P]). In fact, $W^u(x)$ is a C^2 curve which is homeomorphic to $\mathbb{R}$. A measurable decomposition ξ of $\mathbb{T}^2$ is said to be *subordinate to unstable manifolds* if for μ-almost all x in $\mathbb{T}^2$ the following conditions hold:

(F) $\xi(x) \subset W^u(x)$,
(G) $\xi(x)$ contains an open arc of x in $W^u(x)$.

Let $x \in \mathbb{T}^2$ and m_x^u denote the Lebesgue measure of $W^u(x)$. Then a Borel probability measure μ is called an *absolutely continuous conditional measure* on unstable manifolds provided the condition that each μ_x^ξ in a canonical system of conditional measures is absolutely continuous to m_x^u for μ-almost all x in $\mathbb{T}^2$ if ξ is a measurable decomposition that is subordinate to unstable manifolds. It is known (see [S1], [B], [L]) that every Anosov diffeomorphism has a unique SBR measure.

Proof of Proposition B. It is clear that Λ is a f-invariant closed set. Thus, to obtain (a) it suffices to show that Λ is nonempty. Since f is not Anosov, for $\eta > 1$ and $N \geq 1$ there exists $x = x(\eta, N) \in \mathbb{T}^2$ such that for $1 \leq n \leq N$

$$\|Df^n|_{E_x^u}\| \leq \eta.$$

Indeed, if this is false, then we can find $\eta_0 > 1$ and $N_0 \geq 1$ such that for $x \in \mathbb{T}^2$ there exists $1 \leq n \leq N_0$ satisfying $\|Df^n|_{E_x^u}\| > \eta_0$. Thus, for $N \geq 1$ and $x \in \mathbb{T}^2$ there exist n_i $(1 \leq i \leq k)$ and $0 \leq \ell \leq N_0 - 1$ such that $N = n_1 + n_2 + \cdots + n_k + \ell$ and $\|Df^{n_{i+1}}|_{E_{x_i}^u}\| > \eta_0$ for $0 \leq i \leq k-1$ where

$$x_i = \begin{cases} x & (i = 0), \\ f^{n_1+n_2+\cdots+n_i}(x) & (1 \leq i \leq k). \end{cases}$$

Since $\|Df|_{E_y^u}\| \geq 1$ for $y \in \mathbb{T}^2$, we have

$$\begin{aligned} \|Df^N|_{E_x^u}\| &= \prod_0^{k-1} \|Df^{n_{i+1}}|_{E_{x_i}^u}\| \|Df^\ell|_{E_{x_k}^u}\| > (\eta_0)^k \\ &\geq (\eta_0)^{[\frac{N}{N_0}]} \geq (\eta_0)^{\frac{N}{N_0}-1} = (\eta_0)^{-1}\{(\eta_0)^{\frac{1}{N_0}}\}^N. \end{aligned}$$

Put $C = \eta_0^{-1}$ and $\eta_1 = (\eta_0)^{\frac{1}{N_0}}$. Since η_1 and C are independent of x in $\mathbb{T}^2$, f is Anosov. This is a contradiction.

Therefore, for $N \geq 1$ there is $x_N \in \mathbb{T}^2$ satisfying $1 \leq \|Df^n|_{E^u_{x_N}}\| \leq 1 + \frac{1}{N}$ for $1 \leq n \leq 2N+1$. If $f^N(x_N) \to x_0$ (take a subsequence if necessary), then we have that for $i \in \mathbb{Z}$

$$1 \leq \|Df^i|_{E^u_{x_0}}\| = \lim_{N\to\infty} \|Df^i|_{E^u_{f^N(x_N)}}\| \leq \lim_{N\to\infty}\left(1 + \frac{1}{N}\right) = 1,$$

from which Λ is nonempty. (a) is proved.

To show (d), let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denote the natural projection and put $e_j = D\pi(\frac{\partial}{\partial x_j})$ for $j = 1, 2$. Since f is of C^2, we remark that for $i \in \mathbb{Z}$ and $j = 1, 2$, $\varphi_{i,j}(x) = \|D_x f^i(e_j)\|$ is a C^1 function. Choose a sequence $\{\delta_i\}_{i\geq 0}$ of positive numbers satisfying

$$\sum_{i\in\mathbb{Z}} \delta_{|i|} \max\{\varphi_{i,j}(x)^2, \|D_x\varphi_{i,j}\|^2 | x \in \mathbb{T}^2, j = 1, 2\} < \infty,$$

and define a C^1 Riemannian metric $|||\cdot|||$ on $T\mathbb{T}^2$ by

$$|||v|||^2 = \sum_{i\in\mathbb{Z}} \delta_{|i|} \|D_x f^i(v)\|^2 \quad (x \in \mathbb{T}^2,\ v \in T_x\mathbb{T}^2).$$

Then it is easily checked that $|||\cdot|||$ satisfies Proposition B (d).

To show (b) we take a covering map $\bar{\pi} : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\bar{E}^\sigma_x = (D_x\bar{\pi})^{-1}E^\sigma_x$ ($x \in \mathbb{T}^2$, $\sigma = s, u$) is orientable. In fact, $\bar{\pi}$ is 4 to 1. Let $\bar{f} : \mathbb{T}^2 \to \mathbb{T}^2$ be a lifting of f by $\bar{\pi}$. Then we have that for $x \in \mathbb{T}^2$

$$\begin{aligned} &D_x\bar{f}(\bar{E}^\sigma_x) = \bar{E}^\sigma_{\bar{f}(x)} \quad (\sigma = s, u),\\ &\|D_x\bar{f}|_{\bar{E}^s_x}\| \leq \lambda,\\ &\|D_x\bar{f}|_{\bar{E}^u_x}\| \geq 1. \end{aligned}$$

Since $\bar{f}^2 : \mathbb{T}^2 \to \mathbb{T}^2$ preserves an orientation of $\bar{E}^\sigma_x$ ($\sigma = s, u$), for simplicity we replace $\bar{f}^2$ and $\bar{E}^\sigma$ by f and E^σ respectively. Then we can construct a C^0 vector field $X^\sigma : \mathbb{T}^2 \to E^\sigma$ ($\sigma = s, u$) which has no singularities.

From the definition of E^s it follows that X^s is a C^1 vector field. This is checked by using the ideas in the proof of Theorem 6.3 in [H-P]. Thus a C^1 foliation of s-direction, $\mathcal{F}^s$, is constructed. For $x \in \mathbb{T}^2$ denote by $W^s(x)$ a leaf containing x. Then $W^s(x)$ is a C^2 curve which is homeomorphic to $\mathbb{R}$, and has the properties that $T_yW^s(x) = E^s_y$ and $d(f^n(x), f^n(y)) \to 0$ ($n \to \infty$) for $y \in W^s(x)$. Define $W^s_\varepsilon(x) = \{y \in W^s(x) | d_s(x, y) \leq \varepsilon\}$ where d_s denotes the distance between two points along $W^s(x)$. Then we have

$$W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(W^s_\varepsilon(f^n(x))) \quad (x \in \mathbb{T}^2).$$

We need a C^0-foliation of u-direction on $\mathbb{T}^2$ later. To construct it we must use the splitting

$$\|Df|_{E^s_x}\|\|Df^{-1}|_{E^u_{f(x)}}\| \leq \lambda \quad (x \in \mathbb{T}^2)$$

which is called a dominated splitting of $T\mathbb{T}^2$. The splitting is obtained from (iv) in the definition of θ^2. Though the correspondence $x \mapsto E^u_x$ is continuous, by the dominated splitting it is ensured (see [M]) that there exists a family $\{\tilde{W}^u_\varepsilon(x) | x \in \mathbb{T}^2\}$ of C^2 arcs satisfying the conditions:

1. $\tilde{W}^u_\varepsilon(x) \subset B_\varepsilon(x) = \{y \in \mathbb{T}^2 | d(x, y) \leq \varepsilon\}$ for $x \in \mathbb{T}^2$,

2. $T_x\tilde{W}^u_\varepsilon(x) = E^u_x$ for $x \in \mathbb{T}^2$,
3. letting $\tilde{W}^u_{\varepsilon'}(x) = B_{\varepsilon'}(x) \cap \tilde{W}^u_\varepsilon(x)$ for $0 < \varepsilon' \le \varepsilon$, one can find $\varepsilon' > 0$ such that
$$f(\tilde{W}^u_{\varepsilon'}(x)) \subset \tilde{W}^u_\varepsilon(f(x)), \quad f^{-1}(\tilde{W}^u_{\varepsilon'}(x)) \subset \tilde{W}^u_\varepsilon(f^{-1}(x)),$$
4. the correspondence $x \mapsto \tilde{W}^u_\varepsilon(x)$ is continuous with respect to the C^2 metric.

Thus there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $W^s_\varepsilon(x) \cap \tilde{W}^u_\varepsilon(y)$ is one point and $W^s_\varepsilon(x)$ is transverse to $\tilde{W}^u_\varepsilon(y)$. Then we write $[x, y] = W^s_\varepsilon(x) \cap \tilde{W}^u_\varepsilon(y)$.

Lemma 1. *For $x \in \mathbb{T}^2$ the integral curve of X^u through x, γ_x, contains the arc $\tilde{W}^u_\delta(x)$.*

Proof. Suppose $\tilde{W}^u_\delta(x) \not\subset \gamma_x$ for some $x \in \mathbb{T}^2$. Then we can take $y \in B_\delta(x) \cap \gamma_x$ such that $y \notin \tilde{W}^u_\delta(x)$. Thus, $[y, x] = W^s_\varepsilon(y) \cap \tilde{W}^u_\varepsilon(x)$. Since $\|Df^{-1}|_{E^u}\| \le 1$, we have $d(f^{-1}(y), f^{-1}(x)) < \delta$ and then
$$[f^{-1}(y), f^{-1}(x)] = W^s_\varepsilon(f^{-1}(y)) \cap \tilde{W}^u_\varepsilon(f^{-1}(x))$$
and thus $f^{-1}([y, x]) = [f^{-1}(y), f^{-1}(x)]$. Repeating this manner, we have
$$f^{-n}([y, x]) \in W^s_\varepsilon(f^{-n}(y)), \quad f^{-n}([y, x]) \neq f^{-n}(y) \quad (n \ge 0).$$
Using the fact that $\|Df^{-n}|_{E^s}\| \ge \lambda^{-n}$, we have
$$d_s(f^{-n}([y, x]), f^{-n}(y)) \ge \lambda^{-n} d_s([y, x], y).$$
This is a contradiction. □

Since the integral curve γ_x is unique, a C^0 foliation of u-direction, $\mathcal{F}^u$, is constructed. Denote as $W^u(x)$ the leaf containing x in $\mathcal{F}^u$. Then it follows that $T_yW^u(x) = E^u_y$ and $\tilde{W}^u_\varepsilon(y) \subset W^u(x)$ for $y \in W^u(x)$. Remark that each leaf $W^u(x)$ is a C^2 curve.

Lemma 2. *For x, $y \in \mathbb{T}^2$ the cardinality of $W^u(x) \cap W^s(y)$ is infinite.*

Proof. We first prove the lemma when $W^u(x)$ is homeomorphic to $\mathbb{R}$. Since the length of $W^u(x)$, $\ell(W^u(x))$, is infinite, we can find $z_1, z_2 \in W^u(x)$ such that $d(z_1, z_2) < \delta$ and $\tilde{W}^u_\varepsilon(z_1) \cap \tilde{W}^u_\varepsilon(z_2) = \varnothing$.

Since $[z_1, z_2] = W^s_\varepsilon(z_1) \cap \tilde{W}^u_\varepsilon(z_1) \subset W^u(x)$, we denote by γ the closed curve which combine the arc in $W^u(x)$ from $[z_1, z_2]$ to z_1 with the arc in $W^s_\varepsilon(z_1)$ from z_1 to $[z_1, z_2]$. We can suppose that γ is a Jordan closed curve.

If γ is not zero-homotopic, we then obtain the conclusion of Lemma 2. Indeed, let $y \in \mathbb{T}^2$. By the assumption, $\mathbb{T}^2 \setminus \gamma$ is homeomorphic to an annulus. If $W^s(y)$ does not intersect to γ, we then have a contradiction since the existence of a periodic solution of X^s in $\mathbb{T}^2 \setminus \gamma$ is ensured by the Poincaré-Bendixon theorem (Figure 1).

Thus it suffices to prove that γ is not zero-homotopic. If it is false, then there exists a 2-disk D in $\mathbb{T}^2$ such that the boundary of D is equal to γ. Since a C^1 vector field $X^s : D \to E^s$ has no singular points, there is a periodic solution in D by the Poincaré-Bendixon theorem. This contradicts the fact that each leaf in $\mathcal{F}^s$ is homeomorphic to $\mathbb{R}$.

When $W^u(x)$ is a closed curve, we obtain also the conclusion of the lemma. □

Lemma 3. *For $x \in \mathbb{T}^2$, $cl(W^u(x)) = \mathbb{T}^2$ where $cl(E)$ denotes the closure of E.*

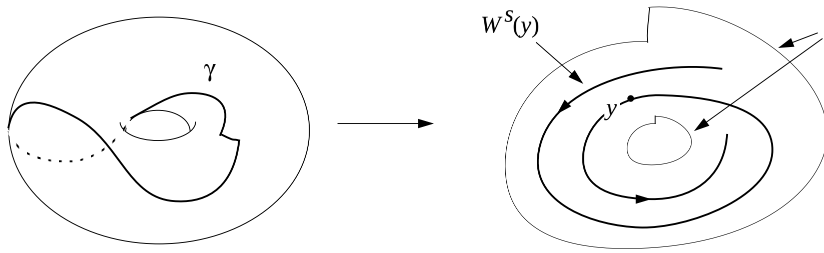

FIGURE 1

Proof. We first prove that $W^u(x)$ is homeomorphic to $\mathbb{R}$. If this is false, then there exists $z \in \mathbb{T}^2$ such that $W^u(z)$ is homeomorphic to a circle. Since $\|Df^{-1}|_{E^u}\| \leq 1$, we have

$$\cdots \leq \ell(W^u(f^{-2}(z))) \leq \ell(W^u(f^{-1}(z))) \leq \ell(W^u(z)).$$

Let $\alpha(z)$ denote the set of α-limit points of z. Take and fix $w \in \alpha(z)$. Then we have $\ell(W^u(f^n(w))) = \ell(W^u(w))$ for $n \in \mathbb{Z}$, and so $W^u(w)$ is contained in Λ.

By Lemma 2 we can take z_1, $z_2 \in W^u(w)$ $(z_1 \neq z_2)$ such that $z_2 \in W^s(z_1)$ and $d_u(z_1, z_2) \leq \delta$ where d_u denotes the distance between two points along $W^u(w)$. Since $W^u_\delta(z_1) \subset \Lambda$, we have $f^n(z_2) \in W^u_\delta(f^n(z_1)) \cap W^s_\delta(f^n(z_1))$ for some $n > 0$, and thus $f^n(z_2) = f^n(z_1)$. This is a contradiction.

Let $x, y \in \mathbb{T}^2$ and let U be a small neighborhood of y. To show the density of $W^u(x)$ it suffices to see that $U \cap W^u(x) \neq \varnothing$. Take a C^2 arc $I \subset W^s_\delta(y) \cap U$. Since $\|Df^{-1}|_{E^s}\| \geq \lambda^{-1}$, we have

$$\ell(f^{-n}(I)) \to \infty \quad (n \to \infty).$$

In a way similar to the proof of Lemma 2, we have $f^{-n}(I) \cap W^u(f^{-n}(x)) \neq \varnothing$ for some $n > 0$ because each leaf of $\mathcal{F}^u$ is homeomorphic to $\mathbb{R}$. Therefore, $U \cap W^u(x) \supset I \cap W^u(x) \neq \varnothing$. □

We remark that Lemma 3 is not true for $W^s(x)$.

Lemma 4. *For $x \in \Lambda$ let I be a C^2 arc containing x such that $I \subset \tilde{W}^u_\delta(x)$. If $\ell(I)$ is positive, then*

$$\sum_{n=0}^{\infty} \ell(f^{-n}(I)) = \infty.$$

Proof. For $\{\tilde{W}^u_\delta(x) | x \in \mathbb{T}^2\}$ a family of C^2 arcs

$$\max_{x \in \mathbb{T}^2} \left| \frac{d^2}{dy^2} \left(f|_{\tilde{W}^u_\delta(x)} \right)(y) \right|$$

is bounded by K from above. Since $f^{-n}(x) \in \Lambda$ and $T_{f^{-n}(x)} \tilde{W}^u_\delta(f^{-n}(x)) = E^u_{f^{-n}(x)}$ for $n \geq 0$, clearly

$$\frac{d}{dy} \left(f|_{\tilde{W}^u_\delta(f^{-n}(x))} \right)(0) = 1.$$

By Taylor's Theorem the graph of $f|_{\tilde{W}^u_\delta(f^{-n}(x))}$ satisfies $\left|\left(f|_{\tilde{W}^u_\delta(f^{-n}(x))}\right)(y)\right| \leq |F(y)|$ ($|y| \leq \delta$, $n \geq 0$) where

$$F(y) = \begin{cases} y + Ky^2 & (y \geq 0), \\ y - Ky^2 & (y < 0). \end{cases}$$

Thus, $\ell(f^{-n}(I)) \geq F^{-n}(\delta)$. Since $\sum_0^\infty F^{-n}(\delta) = \infty$ by ([H-Y] Lemma 4.1), we have $\sum_0^\infty \ell(f^{-n}(I)) = \infty$. □

Since $\mathcal{F}^s$ is a C^1 foliation of $\mathbb{T}^2$, we easily have the following:

Lemma 5. *For $x \in \mathbb{T}^2$ suppose that I and J are C^2 arcs contained in $W^s_\delta(x)$ and $\tilde{W}^u_\delta(x)$ respectively. Then there exists $\kappa > 0$ such that*

$$m([I, J]) \geq \kappa \ell(I)\ell(J)$$

where m is the Lebesgue measure of $\mathbb{T}^2$.

Let $C(x)$ denote the connected component of x in Λ.

Lemma 6. *$C(x) \subset W^u(x)$ for $x \in \Lambda$.*

Proof. If $C(x) \not\subset W^u(x)$ for some $x \in \Lambda$, then there exist $y \in C(x)$ and $\eta > 0$ such that $W^u_\epsilon(y') \cap \Lambda \neq \varnothing$ for $y' \in W^s_\eta(y)$. Take $z \in \alpha(y)$. Then we have $W^s(z) \subset \Lambda$. Since $\|Df|_{E^s_x}\| < \lambda$ and $\|Df|_{E^u_x}\| = 1$ for $x \in \Lambda$, we have $\Lambda \neq \mathbb{T}^2$. Therefore, $\mathrm{cl}\,(W^s(z)) \neq \mathbb{T}^2$.

Fix $w \in \mathbb{T}^2 \setminus \mathrm{cl}\,(W^s(z))$. Let U denote the arcwise connected component of w in $\mathbb{T}^2 \setminus \mathrm{cl}\,(W^s(z))$. Obviously U is open and $f^{-n}(U)$ is the arcwise connected component of $f^{-n}(w)$ in $\mathbb{T}^2 \setminus \mathrm{cl}\,(W^s(z))$ for every $n > 0$. Then we have two cases:

(5) $f^{-n}(U) \cap U = \varnothing$ for all $n > 0$,

(6) $f^{-n_0}(U) = U$ for some $n_0 > 0$.

For (5) we have $\sum_{n=0}^\infty m(f^{-n}(U)) = \infty$. Indeed, from Lemma 3 it follows that the length of the arcwise connected component I of w in $W^u(w) \cap U$ is finite. Let w' be one of the end points of I. Since U is open, w' must belong to $\mathrm{cl}\,(W^s(z)) \subset \Lambda$. Thus Lemmas 4 and 5 ensure that

$$\begin{aligned} \sum_{n=0}^\infty m(f^{-n}(U)) &\geq \sum_{n=0}^\infty m([W^s_\delta(f^{-n}(w')), f^{-n}(I)]) \\ &\geq \sum_{n=0}^\infty \kappa \ell(W^s_\delta(f^{-n}(w')))\ell(f^{-n}(I)) \\ &\geq 2\kappa\delta \sum_{n=0}^\infty \ell(f^{-n}(I)) \\ &= \infty. \end{aligned}$$

When (6) holds, we also have $m(U) = \infty$. Indeed, let I and w' be as above. Since f is orientation preserving, we have $f^{-n_0}(W^s(w')) = W^s(w')$. Thus $f^{-n_0} : W^s(w') \to W^s(w')$ is expanding since $\|Df^{-1}|_{E^s}\| \geq \lambda^{-1}$, and so there exists a unique fixed point $p \in W^s(w')$ of f^{-n_0}. Without loss of generality we suppose that $p \neq w'$. Then w' is not a periodic point. Thus, for $r > 0$ small enough we have

$$[W^s_r(f^{-kn_0}(w')), f^{-kn_0}(I)] \cap [W^s_r(f^{-k'n_0}(w')), f^{-k'n_0}(I)] = \varnothing \quad (k \neq k'),$$

and thus by Lemmas 4 and 5

$$\begin{aligned} m(U) &\geq m\left(\bigcup_{k=0}^{\infty}[W_r^s(f^{-kn_0}(w')), f^{-kn_0}(I)]\right) \\ &= \sum_{k=0}^{\infty} m([W_r^s(f^{-kn_0}(w')), f^{-kn_0}(I)]) \\ &\geq \sum_{k=0}^{\infty} \kappa\ell(W_r^s(f^{-kn_0}(w')))\ell(f^{-kn_0}(I)) \\ &\geq 2\kappa r\sum_{k=0}^{\infty}\ell(f^{-kn_0}(I)) = \infty. \end{aligned}$$

In any case we have $m(\mathbb{T}^2) = \infty$. This is a contradiction. □

Lemma 6 tells us that $C(x)$ is either a single point, or a C^2 arc in $W^u(x)$. We remark that $\ell(C(x))$ is finite. This follows from the fact if $\ell(C(x))$ is infinite then $\mathrm{cl}(C(x)) = \mathbb{T}^2$ by Lemma 3. Therefore the second statement of Proposition B (b) was proved.

Since $\|Df|_{E_y^u}\| = 1$ for $y \in C(x)$, the length of $C(x)$ is f-invariant. Then, using the next lemma, it follows that $f^{m(x)}|_{C(x)}$ is the identity map of $C(x)$. This implies Proposition B (c).

Lemma 7. *For $x \in \Lambda$ there exists $m = m(x) > 0$ such that $f^m(C(x)) = C(x)$.*

Proof. We first prove the lemma for the case when $C(x)$ is a C^2 arc. To see so let y be one of the end points of $C(x)$ in $W^u(x)$. It is clear that $\alpha(y) \subset \Lambda$. If we establish that $\alpha(y)$ is finite, then each element belonging to $\alpha(y)$ is periodic. Thus there is $z \in \alpha(y)$ such that $y \in W^u(z)$. Then we have $y = z$. Indeed, if $y \neq z$ and $f^{-i}(z) = z$, then we have that for $k > 0$

$$d_u(y, z) = d_u(y, f^{-i}(y)) + d_u(f^{-i}(y), f^{-2i}(y)) + \cdots + d_u(f^{-ki}(y), z).$$

Thus we have $d_u(y, z) = \infty$ by Lemma 4 since k is arbitrary. This is a contradiction. Therefore y is periodic. Let m be the period of y by f. Then $f^m(C(x)) = C(x)$ since y is an end point of $C(x)$, and since the length of $C(x)$ is f-invariant. Thus it suffices to prove that $\alpha(y)$ is finite.

We first prove that $\alpha(y)$ is totally disconnected. If this is false, then there exists $z \in \alpha(y)$ such that $\tilde{W}_\eta^u(z) \subset \alpha(y)$ for small $\eta > 0$. Choose an increasing sequence $\{n_k\}$ such that $\{f^{-n_k}(y)\}$ converges to z as $k \to \infty$. Then $[f^{-n_k}(y), z]$ converges to z as $k \to \infty$. Since

$$\begin{aligned} d(y, f^{n_k}(z)) &\leq d_s(y, f^{n_k}([f^{-n_k}(y), z])) + d_u(f^{n_k}([f^{-n_k}(y), z]), f^{n_k}(z)) \\ &\leq \lambda^{n_k} d_s(f^{-n_k}(y), [f^{-n_k}(y), z]) + d_u([f^{-n_k}(y), z], z) \\ &\to 0 \quad (k \to \infty), \end{aligned}$$

$f^{n_k}(\tilde{W}_\eta^u(z))$ converges to $\tilde{W}_\eta^u(y)$ under the Hausdorff topology. Remark that $f^{n_k}(\tilde{W}_\eta^u(z)) \subset \Lambda$ for $k > 0$ (and then $\tilde{W}_\eta^u(y) \subset \Lambda$). Then we have $\tilde{W}_\eta^u(y) \subset C(x)$, thus contradicting the choice of y. Therefore $\alpha(y)$ is totally disconnected.

Suppose that $\alpha(y)$ is infinite to obtain a contradiction. For $w \in \mathbb{T}^2$, $\tilde{W}_\epsilon^u(w) \setminus \{w\}$ is expressed as the union $\tilde{W}_\epsilon^u(w) \setminus \{w\} = I_w^1 \cup I_w^2$ of C^2 arcs I_w^1 and I_w^2 in $W^u(w)$

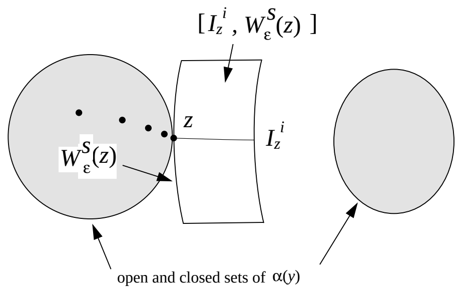

FIGURE 2

with $I_w^1 \cap I_w^2 = \varnothing$. Since the collection of open closed subsets of $\alpha(y)$ is a base of $\alpha(y)$, there exist a non-periodic point $z \in \alpha(y)$ and i (= 1 or 2) such that

$$\text{int}([I_z^i, W_\epsilon^s(z)]) \cap \alpha(y) = \varnothing \tag{7}$$

for $\varepsilon > 0$ small enough (Figure 2). Indeed, if z is a periodic point, then we have $y \notin W^u(z)$ since $\alpha(y)$ and the orbit of z agree when $y \in W^u(z)$, and since $\alpha(y)$ is infinite by the assumption. Thus $\{\alpha(y) \cap W^s(z)\} \setminus \{z\}$ is nonempty, then we can take a non-periodic point from the set. Notice that $f^{-n}(z) \notin W_\epsilon^s(z)$ $(n > 0)$ for sufficiently small $\epsilon > 0$. For simplicity put $I_z = I_z^i$.

Take and fix $\tau > 0$ small enough. We define $R_n = [f^{-n}(I_z), W_\tau^s(z)]$ for $n \geq 0$. Then, $R_{n+1} \subset f^{-1}(R_n)$ for all $n \geq 0$, and $R_n \cap R_m = \varnothing$ for all $n \neq m$. Indeed, suppose $R_m \cap R_n \neq \varnothing$ for some $m > n \geq 0$. Put $k = m - n$. Since

$$\varnothing \neq f^n(R_m \cap R_n) \subset f^n(f^{-n}(R_k) \cap f^{-n}(R_0)) = R_k \cap R_0,$$

we have that $z \in \text{int}([f^{-k}(I_z), W_\epsilon^s(f^{-k}(z))])$ since $f^{-k}(z) \notin W_\epsilon^s(z)$. Therefore, $f^k(z) \in \text{int}([I_z, W_\epsilon^s(z)])$, thus contradicting (7).

Since $R_n \cap R_m = \varnothing$ for all $n \neq m$, by Lemmas 4 and 5,

$$\begin{aligned} m\left(\bigcup_{n\geq 0} R_n\right) &= \sum_{n\geq 0} m(R_n) \\ &\geq \sum_{n\geq 0} \kappa\ell(f^{-n}(I_z))\ell(W_\tau^s(z)) \\ &\geq 2\kappa\tau \sum_{n\geq 0} \ell(f^{-n}(I_z)) \\ &= \infty. \end{aligned}$$

But this is impossible. Therefore $\alpha(x)$ is finite.

When $C(x)$ is a single point, we also obtain the conclusion of the lemma. □

To complete the proof of Proposition B it suffices to show that Λ splits into the union of finite connected sets. To obtain it suppose the cardinality of $\{C(x) | x \in \Lambda\}$ is infinite. If $\{x_i\}$ is an infinite sequence in Λ and $x_i \to x$ as $i \to \infty$, then x is also a periodic point by Proposition B (c).

If $x_i \in W^u(x)$ for some i, then x and x_i are joined by a C^2 arc I in $W^u(x)$. Since $\|Df^{-1}|_{E^u}\| \leq 1$, we have $\ell(f^{-n}(I)) \leq \ell(I)$ for $n > 0$. Thus, $\ell(f^n(I)) = \ell(I)$ for all $n \in \mathbb{Z}$ because x_i and x are periodic points. This implies that $C(x_i) = C(x)$.

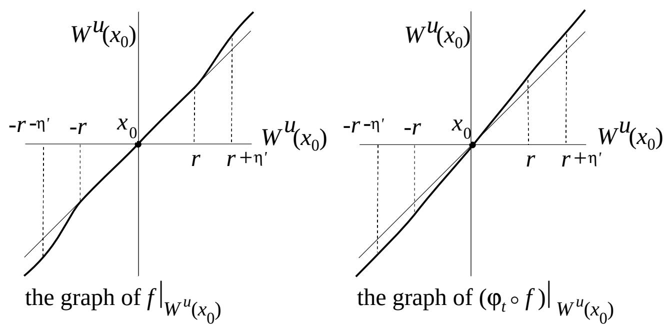

FIGURE 3

Since the cardinality of $\{C(x)|x \in \Lambda\}$ is infinite, we may assume that $x_i \notin W^u(x)$ for all i.

Since $x_i \notin W^u(x)$ for all $i \geq 0$, we can find $y \in W^s(x)$ such that $y \in \Lambda$ and $y \neq x$. Then $y \in \Lambda$ is not a periodic point. This contradicts Proposition B (c). Therefore Λ is expressed as the union of finite connected sets.

Proof of Proposition C. Let Λ be as in Proposition B. We give the proof of Proposition C for the case when Λ is a C^2 arc. When Λ is a single point, or in general, then the conclusion is obtained in a similar argument.

For $x \in \mathbb{T}^2$ and $r > 0$ we define $W^u_r(x) = \{y \in W^u(x)|d_u(x,y) \leq r\}$. Since Λ is a C^2 arc which is tangential to E^u, Λ is expressed as $\Lambda = W^u_r(x_0)$ for some $x_0 \in \Lambda$ and $r > 0$. For $\eta > 0$ small enough define

$$R_\eta = \bigcup_{x \in W^u_{r+\eta}(x_0)} W^s_\eta(x).$$

Since $\Lambda \subset \mathrm{int}R_\eta$, by Proposition B (d) there exists $\varepsilon > 0$ such that

$$|||D_x f|_{E^u}||| > e^{\varepsilon} \quad (x \notin R_\eta) \tag{8}$$

where $|||\cdot|||$ is the norm as in Proposition B. A projection $\pi^u : R_\eta \to W^u_{r+\eta}(x_0)$ defined by $\{\pi^u(x)\} = W^s_\eta(x) \cap W^u_{r+\eta}(x_0)$ for $x \in R_\eta$ is a C^1 map, since $\mathcal{F}^s$ is a C^1 foliation of $\mathbb{T}^2$.

Since $|||D_x\pi^u|_{E^u}||| = 1$ for $x \in \Lambda$, we have that for $0 < \eta'$ $(< \eta)$ small enough

$$e^{-\varepsilon/5} \leq |||D_x\pi^u|_{E^u}||| \leq e^{\varepsilon/5} \quad (x \in R_{\eta'}). \tag{9}$$

Then we can construct a one-parameter family $\varphi_t : \mathbb{T}^2 \to \mathbb{T}^2$ $(t \in [0,1])$ which satisfies the following:

(10) φ_t is a C^2 diffeomorphism for $t \in [0,1]$,
(11) $d_2(\varphi_t, \mathrm{id}) \to 0$ as $t \to 0$ where d_2 and id denote a C^2 metric and the identity map respectively,
(12) $\varphi_t(x) = x$ for $t \in [0,1]$ and $x \notin R_{\eta'}$,
(13) for $t \in (0,1]$, $\varphi_t(W^u_{r+\eta'}(x_0)) = W^u_{r+\eta'}(x_0)$ and $|||D_x(\varphi_t \circ f)|_{E^u}||| > 1$ $(x \in W^u_{r+\eta'}(x_0))$ (Figure 3).

Put $f_t = \varphi_t \circ f$ for $t \in [0,1]$. Clearly f_t is a C^2 diffeomorphism for $t \in [0,1]$ (by (10)) and f is approximated by $\{f_t\}$ with respect to the C^2 topology (by (11)).

The remainder of the proof is only to show that f_t is Anosov for small $t \in (0,1]$. By (8) and Proposition B, f satisfies

$$\begin{cases} |||D_xf|_{E^s}|||\ |||D_{f(x)}f^{-1}|_{E^u}||| \le \lambda & (x \in \mathbb{T}^2), \\ |||D_xf|_{E^s}||| \le \lambda & (x \in \mathbb{T}^2), \\ |||D_xf|_{E^u}||| > e^{\varepsilon} & (x \notin R_\eta), \\ |||D_xf|_{E^u}||| > 1 & (x \notin R_{\eta'}). \end{cases}$$

Then we can choose $t_0 > 0$ such that for $0 < t < t_0$ there exist $\lambda < \lambda' < 1$ and a Df_t invariant splitting $T_x\mathbb{T}^2 = E^u_x(t) \oplus E^s_x(t)$ $(x \in \mathbb{T}^2)$ satisfying

$$\begin{cases} |||D_xf_t|_{E^s(t)}|||\ |||D_{f(x)}f_t^{-1}|_{E^u(t)}||| \le \lambda' & (x \in \mathbb{T}^2), \\ |||D_xf_t|_{E^s(t)}||| \le \lambda' & (x \in \mathbb{T}^2), \\ |||D_xf_t|_{E^u(t)}||| > e^{\varepsilon/2} & (x \notin R_\eta), \\ |||D_xf_t|_{E^u(t)}||| > 1 & (x \notin R_{\eta'}). \end{cases}$$

Fix t with $0 < t < t_0$. In the same manner as the proof of Proposition B construct a C^1 foliation $\mathcal{F}^s_t = \{W^s(x,t)\}_{x\in\mathbb{T}^2}$ which is tangential to $E^s(t)$. Denote by $W^s_\eta(x,t)$ the set of $y \in W^s(x,t)$ such that $d_s(x,y) \le \eta$, and define a projection $\pi^u_t : R_{\eta'} \to W^u_{r+\eta}(x_0)$ by

$$\{\pi^u_t(x)\} = W^s_\eta(x,t) \cap W^u_{r+\eta}(x_0) \quad (x \in R_{\eta'}).$$

By (9) we can assume that for every $x \in R_{\eta'}$

$$e^{-\varepsilon/4} \le |||D_x\pi^u_t|_{E^u(t)}||| \le e^{\varepsilon/4} \tag{14}$$

(by taking t_0 sufficiently small if necessary).

Put $\mu_t = \inf\{|||D_xf_t|_{E^u(t)}|||\ |x \in W^u_{r+\eta}(x_0)\}$. Then, $\mu_t > 1$ by (12) and (13). If $f^i(x) \in R_{\eta'}$ for $0 \le i \le n$, then we have $|||D_xf^n_t|_{E^u(t)}||| \ge e^{-\varepsilon/2}\mu^n_t$. This follows from the fact that by (14)

$$\begin{aligned} |||D_xf^n_t|_{E^u(t)}||| \ge& |||(D_{f^n_t(x)}\pi^u_t|_{E^u(t)})^{-1}|||\ |||D_{\pi^u_t(x)}f^n_t|_{E^u(t)}|||\ |||D_x\pi^u_t|_{E^u(t)}||| \\ \ge& e^{-\varepsilon/4}\mu^n_t e^{-\varepsilon/4} = e^{-\varepsilon/2}\mu^n_t. \end{aligned}$$

Let K_1 be a large number such that for $x \in R_{\eta'}$ and g $(= f$ or $f^{-1})$ if $g(x) \notin R_{\eta'}$ then $g^{K_1}(x) \notin R_\eta$, and K_2 be a positive integer satisfying $e^{-\varepsilon/2} \cdot \mu_t^{K_2} > 1$. Put $K = 2K_1 + K_2$. Then we have $|||D_xf^K_t|_{E^u(t)}||| > 1$ for $x \in \mathbb{T}^2$. Therefore it follows that f_t is Anosov by the technique of the proof of Proposition B (a).

Proof of Proposition D. For the proof we need Proposition B and the technique used in [L-Y]. Suppose the proposition is false. For $\varepsilon > 0$ we have (see [L-Y], Lemma 3.2 and Corollary 6.2) that there is a Borel set S such that the following conditions hold:

(15) $\mu(S) > 0$.

(16) There is a family $\{D_\alpha\}$ of C^2 arcs satisfying

(16-1) $D_\alpha \cap D_{\alpha'} = \varnothing \quad (\alpha \ne \alpha')$,

(16-2) $S = \bigcup_\alpha D_\alpha$,

(16-3) if $x \in D_\alpha$, then $D_\alpha \subset \tilde{W}^u_\varepsilon(x)$ and D_α is open in $\tilde{W}^u_\varepsilon(x)$.

(17) Let $\{\mu_\alpha\}$ be a canonical system of conditional Borel probability measures, then each μ_α is absolutely continuous with respect to the Lebesgue measure

m_α of D_α, and if $\rho_\alpha : D_\alpha \to \mathbb{R}^+$ is the density function of m_α, then there is $L > 0$ such that for $(x, y) \in D_\alpha \times D_\alpha$

$$\left| \log \frac{\rho_\alpha(y)}{\rho_\alpha(x)} \right| \leq L d_u(x, y).$$

By (16-3) and (17) we have

$$\left| \frac{\rho_\alpha(x)}{\rho_\alpha(y)} \right| \leq e^{L\varepsilon}. \tag{18}$$

Since μ is a SBR measure of $\mathbb{T}^2$ by the assumption, μ-almost all x has a positive Lyapunov exponent. Then, for μ-a.e. x, E^u_x is the subspace corresponding to the exponent. Since $f \in \theta^2 \setminus A(\mathbb{T}^2)$, we have $\|Df|_{E^u}\| \geq 1$ and thus $\ell(f^n D_\alpha) \geq \ell(D_\alpha)$. If a point in D_α has a positive Lyapunov exponent, then $\ell(f^n D_\alpha) \nearrow \infty$ as $n \to \infty$. Therefore, without loss of generality we can suppose that $\ell(f^n D_\alpha) \nearrow \infty$ for all α.

Let Λ be as in Proposition B and take $z \in \Lambda$. By Proposition B, z is a periodic point of f. For simplicity we assume that z is a fixed point. By Lemma 3 there exists $n(\alpha) = n \geq 0$ such that

$$f^n(D_\alpha) \cap W^s(z) \neq \varnothing.$$

Thus we have

$$D_\alpha \cap W^s(z) = f^{-n}(f^n(D_\alpha)) \cap W^s(z) \neq \varnothing.$$

Since D_α contains a point with the positive Lyapunov exponent, we can find a compact subset $C \subset W^s(z)$ such that

$$\mu\left(\bigcup_\alpha \{D_\alpha | D_\alpha \cap C \neq \varnothing\} \right) > 0. \tag{19}$$

Choose $k > 0$ such that $d_s(f^k(y), z)$ is small enough for all $y \in C$, and put $C' = f^k(C)$ for simplicity. Without loss of generality we can suppose

$$[\tilde{W}^u_\tau(z), C'] \subset \bigcup_\alpha f^k(D_\alpha) \tag{20}$$

for $\tau > 0$ small enough. Define $R_n = [\tilde{W}^u_\tau(z), f^n(C')]$ for $n \geq 0$. Then we have that for $n \neq m$, $R_n \cap R_m = \varnothing$ by taking the size of C' sufficiently small. We remark that there is $\eta > 0$ such that for $y \in \tilde{W}^u_\tau(z)$ and $w \in C'$

$$d_u([y, w], w) \geq \eta d_u(y, z) \tag{21}$$

since $\mathcal{F}^s$ is a C^1 foliation of $\mathbb{T}^2$.

Put $S_0 = \bigcup_\alpha f^k(D_\alpha)$. Then S_0 is a Borel set satisfying the conditions (15), (16) and (17). Thus the μ-values of $[\tilde{W}^u_\tau(z), C']$ are positive by (18), (19) and (20). Since $\{[\tilde{W}^u_\tau(z), y] | y \in C'\}$ is a decomposition of $S_1 = [\tilde{W}^u_\tau(z), C']$, S_1 is a Borel set satisfying (15), (16) and (17). Thus we have that

$$\begin{aligned}\sum_n \mu(R_n) &= \sum_n \mu(f^{-n}(R_n)) \\ &= \sum_n \mu(f^{-n}([\tilde{W}^u_\tau(z), f^n(C')])) \\ &= \sum_n \mu([f^{-n}(\tilde{W}^u_\tau(z)), C']) \\ &\geq \sum_n e^{-L\varepsilon}\eta\ell(f^{-n}(\tilde{W}^u_\tau(z))) \quad \text{(by (18) and (21))}.\end{aligned}$$

Since $\mu(R_1) > 0$, by applying Lemma 4 we have $\mu(\bigcup R_n) = \infty$. This is a contradiction.

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